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COMMENT

Some observations on the nature of solutions for the interaction $V(x) = x^2 + \lambda x^2/(1 + gx^2)$

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Abstract. We study the nature of exact solutions for the non-polynomial interaction. In particular it is shown that all the solutions are supersymmetric. It is also shown that this problem admits a dynamical SU(2) symmetry so that an arbitrarily large part of the spectrum, but not the whole spectrum, can be determined exactly.

In a number of recent papers [1-4] several authors studied exact solutions for the non-polynomial interaction. Exact solutions for this potential have also been studied [5] using supersymmetry. By comparing the solutions obtained in [1-4] with those of [5] it can be easily seen that the solutions in both cases are the same. The question here is why apparently non-supersymmetric methods lead to supersymmetric results. Our purpose is to clarify this issue. We shall also analyse the problem from a different angle: it will be shown that this problem admits partial algebraisation [6, 7] so that an arbitrarily large part of the spectrum can always be determined although the spectrum cannot be found entirely.

To show that the methods used in [1-4] yield exact eigenvalues and eigenfunctions only when they become supersymmetric, we consider the most general of these methods [4]. Here one considers eigensolutions (we consider even parity solutions) of the form:

$$Y = \sum_{i=0}^{n} a_i x^{2i}(1 + gx^2)\exp(-x^2/2). \tag{1}$$

Next we put (1) into the Schrödinger equation

$$\frac{d^2y}{dx^2} + (E - V)y = 0 \tag{2}$$

$$V = x^2 + \frac{\lambda x^2}{(1 + gx^2)} \tag{3}$$

and form a recurrence relation involving the coefficients a_i. The energy eigenvalues are determined from these relations. To see the connection with SUSY we note that (1) can be written as

$$Y = (1 + gx^2)\exp(-x^2/2)f(x) \tag{4}$$

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where $f(x)$ is given by

$$f(x)=\sum_{i=0}^{n} a_i x^{2i}. \tag{5}$$

Now we also have

$$y(x)=\exp\left(-\int^{x}\left(t-\frac{2gt}{1+gt^2}-\frac{f'(t)}{f(t)}\right)\right). \tag{6}$$

It is now easy to recognise (6) as the ground-state wavefunction of a SUSY system with superpotential given by

$$W(x)=x-\frac{2gx}{(1+gx^2)}-\frac{f'(x)}{f(x)}. \tag{7}$$

The potential of the supersymmetric system is given by

$$V_{\pm}(x)=W^2(x)\pm W'(x). \tag{8}$$

Next we note that $f(x)$ is a polynomial of degree n in x^2 and thus we can write

$$f(x)=a_n(x^2-\alpha_n)(x^2-\alpha_{n-1})\ldots(x^2-\alpha_0) \tag{9}$$

so that

$$\frac{f'(x)}{f(x)}=\sum_{i=0}^{n}\frac{2x}{(x^2-\alpha_i)}. \tag{10}$$

Hence the complete superpotential is given by

$$W(x)=x-\frac{2gx}{(1+gx^2)}-\sum_{i=0}^{n}\frac{2x}{(x^2-\alpha_i)}. \tag{11}$$

The next step is to calculate $W^2(x)-W'(x)$, i.e. $V_-(x)$ and identify it with (3); this identification will induce a constant with a relation between λ and g of the form, $-\lambda/g=F(g)$. The form of $F(g)$ will depend on the number of terms on the RHS of (10). The number of values each α_i can assume will determine the number of solutions for the same potential.

For example in the absence of the term $f'(x)/f(x)$ on the RHS of (7)

$$F(g)=(6g+4) \tag{12}$$

([5] equation (31); odd parity solution) and if we take only one term for $f'(x)/f(x)$ (i.e. $n=1$ in (10)) then the even parity solution gives

$$F(g)=7g+6\pm\sqrt{25g^2-12+4}. \tag{13}$$

For $g=2/3$, (12) and (13), with the sign before the square root are equal and we get two solutions, namely

$$\varphi_1(x)=x(1+\tfrac{2}{3}x^2)\,\mathrm{e}^{-x^2/2} \tag{14}$$

and

$$\varphi_1(x)=(1+\tfrac{2}{3}x^2)(1-\tfrac{2}{3}x^2)\,\mathrm{e}^{-x^2/2} \tag{15}$$

with eigenvalues -1 and 1 respectively and $\lambda=-16/3$ and $g=2/3$; this solution has also been obtained by Lakhtakia. For other solution see [5].

Now the question is why only a finite part of the spectrum can be found. The answer lies in the fact that this problem admits a dynamical SU(2) symmetry which allows determination of an arbitrary but finite part of the spectrum. To show this we note that (4) can be written as

$$Y(x)=\exp\left(\int\left(x-\frac{2gx}{1+gx^2}\right)\mathrm{d}x\right)\tilde{Y}(x) \tag{16}$$

and (12) can easily be recognised as an imaginary gauge transformation on $Y(x)$ with gauge function given by

$$A(x)=x-\frac{2gx}{(1+gx^2)}. \tag{17}$$

Substituting (16) in equation (2) we find

$$H_G Y(x)=E\tilde{Y}(x) \tag{18}$$

where H_G is the gauge Hamiltonian and is given by

$$H_G=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+A(x)\frac{\mathrm{d}}{\mathrm{d}x}+\Delta V \tag{19}$$

$$\Delta V=V(x)+A'(x)-A^2(x). \tag{20}$$

It is now necessary to write the gauge Hamiltonian (19) in terms of the generators of the SU(2) group. These generators, in the form of differential operators, are given by

$$T^+=2j\xi-\frac{\mathrm{d}}{\mathrm{d}\xi} \tag{21}$$

$$T^0=-j+\xi\frac{\mathrm{d}}{\mathrm{d}\xi} \tag{22}$$

$$T^-=\frac{\mathrm{d}}{\mathrm{d}\xi} \tag{23}$$

where j denotes the spin of the representation and can assume semi-integer values. The dimension of the representation is $(2j+1)$. Corresponding to the generators (20)-(22) the finite-dimensional representation of the group is

$$R_j=\{1,\xi,\xi^2,\ldots,\xi^{2j}\}. \tag{24}$$

However it is not possible to express H_G in terms of the generators; instead we express $(1+gx^2)H_G$ in terms of $T^{\pm}$ and T^0 and the eigenvalues can be found from the following relation [8, 9]:

$$(1+gx^2)(H_G-E)\tilde{Y}=0. \tag{25}$$

The expression for $(1+gx^2)H_G$ in terms of $T^{\pm}$, T^0 is

$$(1+gx^2)H_G=\sum_{a=0,-}T^aT^0+\sum_{a=\pm}C_aT^a+\text{constant}. \tag{26}$$

It is clear that the above expression operates on polynomials in $\xi(=gx^2)$ of degree $2j$. Next we choose the basis as follows:

$$\tilde{y}=\{1,\xi,\xi^2,\ldots,\xi^{2j},\tilde{y}_{2j+2},\tilde{y}_{2j+3},\ldots\} \tag{27}$$

where $\tilde{y}_{2j+2}$ etc are an arbitrary set of functions orthogonal to $1, \xi, \ldots, \xi^{2j}$ with weight function $\int A\,dx$. It is then clear that the energy matrix is split into two parts: the upper left block is a $(2j+1)\times(2j+1)$ matrix while the one in the bottom right-hand corner is infinite dimensional. One can now diagonalise the finite $(2j+1)\times(2j+1)$ to obtain the $(2j+1)$ energy eigenvalues. The corresponding wavefunctions can be obtained from (16) and the fact that y can be written as

$$\tilde{y}(x)=(gx^2-a_1)(gx^2-a_2)\ldots(gx^2-a_{2j}). \tag{28}$$

For the details regarding determination of the parameters a_i we refer the reader to [6].

In conclusion it has been shown that the power series method of solving the non-polynomial interaction [1-4] is supersymmetric and naturally this method leads to SUSY solutions [5]. The other result which is totally new is that this problem has, apart from SUSY, a dynamical SU(2) symmetry which allows partial algebraisations of the problem i.e. a finite part of the spectrum can always be exactly determined.

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